ANALYSIS OF UNBIASED ESTIMATORS USING GEOMETRIC FAILURE DATA

HERMAN C. QUITMEYER

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Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
with major
in
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ABSTRACT

An important example of an event which obeys the geometric probability law with parameter Θ is the number of cycles required to obtain the first failure of an item, where the success of each cycle is independent with probability of success Θ.

The true probability of success, in manufactured items, is usually unknown and must be estimated on the basis of observed data obtained from a test of sample items. An important estimator for this probability is the unbiased estimator, defined as that estimator whose expected value equals the true value of the parameter. In this study, an unbiased estimator for Θ is derived. This estimator is based on the results of a series of independent items, each cycled to first failure.

Series approximations for the variance of this estimator are derived, and some values of the variance are tabled for those cases thought to be of special interest.

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TABLE OF SYMBOLS AND ABBREVIATIONS

Symbol	Description
θ	Probability of success of a single cycle
n	Number of items cycled till first failure
s	Total successful cycles completed in n tests
N_s	The number of ways n tests can sum to s successes
θ_s	The unbiased estimator for θ
M_s	The product $(N_s \theta_s)$
R	The ratio $\frac{(1-\theta)}{\theta}$

1. Introduction.

The geometric probability law with parameter θ, where $0 \leq \theta \leq 1$, is specified by the probability mass function:

$$p(x) = \theta^x (1 - \theta) \quad \text{for } x = 0, 1, 2, \ldots.$$
$$= 0 \quad \text{Otherwise,}$$

where x is the number of successful cycles to first failure of one item.

When n independent items are cycled to first failure, the probability of each ordered event is

$$\prod_{i=1}^{n} \theta^{x_i}(1-\theta) = \theta^s (1-\theta)^n \quad \text{where } s = \sum x_i .$$

The probability of exactly s successes is thus the union of all events with exactly s successes. The number of such events is the number of ways n tests can sum to s successes. Therefore,

$$p(s) = N_s \theta^s (1-\theta)^n \quad \text{for } s = 0, 1, 2, 3, \ldots\ldots$$
$$= 0 \quad \text{otherwise.}$$

Since N, M, and $\hat{\theta}$ are functions of both n and s, the subscript (n,s) is necessary to uniquely identify the value. However, since n is a known variable which is fixed prior to testing, the abbreviated subscript s will be used.

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2. Derivation of the Unbiased Estimator.

By definition, and estimator $\hat{\theta}_s$ is unbiased if its expected value equals the true value of θ, where the expected value $E(\hat{\theta}_s)$ is defined by

$$E(\hat{\theta}_s) = \sum_s \hat{\theta}_s \, p(s) \quad \text{over all s such that } p(s) > 0.$$

Thus, $\hat{\theta}_s$ is an unbiased estimator for θ if and only if

$$\theta = \sum_{s=0}^{\infty} \hat{\theta}_s \, N_s \, \theta^s \, (1-\theta)^n$$

$$= \sum_{s=0}^{\infty} M_s \, \theta^s \, (1-\theta)^n \, .$$

By expanding the $(1-\theta)^n$ term, the summation becomes

$$\theta = \sum_{s=0}^{\infty} M_s \, \theta^s \left[1 - \binom{n}{1}\theta + \binom{n}{2}\theta^2 - - - - +(-1)^{n-1}\binom{n}{n-1}\theta^{n-1} + (-1)^n \, \theta^n \right]$$

$$= M_o \left[1 - \binom{n}{1}\theta + \binom{n}{2}\theta^2 - - - - - +(-1)^{n-1}\binom{n}{n-1}\theta^{n-1} + (-1)^n \, \theta^n \right]$$

$$+ M_1 \left[\theta - \binom{n}{1}\theta^2 + \binom{n}{2}\theta^3 - - - - - +(-1)^{n-1}\binom{n}{n-1}\theta^{n} + (-1)^n \, \theta^{n+1} \right]$$

$$+ M_2 \left[\theta^2 - \binom{n}{1}\theta^3 + \binom{n}{2}\theta^4 - - - - - +(-1)^{n-1}\binom{n}{n-1}\theta^{n+1} + (-1)^n \, \theta^{n+2} \right]$$

$$+ - - - - - - - \text{ etc.}$$

Rearranging terms as coefficients of a power series in θ, we obtain

$$\theta = \left[M_o \right]$$

$$+ \theta \left[M_1 - \binom{n}{1} M_o \right]$$

$$+ \theta^2 \left[M_2 - \binom{n}{1} M_1 + \binom{n}{2} M_o \right]$$

$$+ \theta^3 \left[M_3 - \binom{n}{1} M_2 + \binom{n}{2} M_1 - \binom{n}{3} M_o \right]$$

$+ \quad - - - \quad$ etc., until the θ^nth term. Then,

$$+ \theta^{n} \left[M_n - \binom{n}{1} M_{n-1} + - - + (-1)^{n-1} \binom{n}{n-1} M_1 + (-1)^n M_o \right]$$

$$+ \theta^{n+1} \left[M_{n+1} - \binom{n}{1} M_n + - - + (-1)^{n-1} \binom{n}{n-1} M_2 + (-1)^n M_1 \right]$$

$$+ \theta^{n+2} \left[M_{n+2} - \binom{n}{1} M_{n+1} + - - + (-1)^{n-1} \binom{n}{n-1} M_3 + (-1)^n M_2 \right]$$

$$+ \; - - - - - \; \text{etc.}$$

By equating coefficients: $\left[M_o \right] = 0$

$$\left[M_1 - \binom{n}{1} M_o \right] = 1.$$

All remaining coefficients are zero.

Solving for M_s $(s = 0, 1, 2, ---)$, the following table of equations is produced:

$$\begin{aligned}
M_o &= 0 \\
M_1 &= 1 \\
M_2 &= \binom{n}{1} M_1 \\
M_3 &= \binom{n}{1} M_2 - \binom{n}{2} M_1 \\
M_4 &= \binom{n}{1} M_3 - \binom{n}{2} M_2 + \binom{n}{3} M_1 \\
M_5 &= \binom{n}{1} M_4 - \binom{n}{2} M_3 + \binom{n}{3} M_2 - \binom{n}{4} M_1 \\
M_6 &= \binom{n}{1} M_5 - \binom{n}{2} M_4 + \binom{n}{3} M_3 - \binom{n}{4} M_2 + \binom{n}{5} M_1 \\
M_7 &= \binom{n}{1} M_6 - \binom{n}{2} M_5 + \binom{n}{3} M_4 - \binom{n}{4} M_3 + \binom{n}{5} M_2 - \binom{n}{6} M_1 \\
M_8 &= \binom{n}{1} M_7 - \binom{n}{2} M_6 + \binom{n}{3} M_5 - \binom{n}{4} M_4 + \binom{n}{5} M_3 - \binom{n}{6} M_2 + \binom{n}{7} M_1 \\
&\;\;\vdots
\end{aligned} \qquad (1)$$

Example of table with $n = 5$.

and so forth.

Initial numerical solutions for M_s are as follows:

$$M_o = 0$$

$$M_1 = 1$$

$$M_2 = n$$

$$M_3 = \binom{n}{1} n - \binom{n}{2} = \frac{n(n+1)}{2!} = \binom{n+1}{2}$$

$$M_4 = \binom{n}{1}\binom{n+1}{2} - \binom{n}{2} n + \binom{n}{3} = \frac{n(n+1)(n+2)}{3!} = \binom{n+2}{3} .$$

Thus, M_s can be written as the expression

$$M_s = \binom{n+s-2}{s-1} \quad \text{for} \quad s = 0, 1, 2, 3, 4.$$

The proof that this expression holds for all s will be by induction.

The assumption: $M_i = \binom{n+i-2}{i-1}$ for $i = 0, 1, 2, 3, ---, (k-1)$.

To prove: $M_k = \binom{n+k-2}{k-1}$.

From equations (1), M_k can be written as the following summation of the M_i:

$$M_k = \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i}\binom{n+k-i-2}{k-i-1} = -\left[\sum_{i=1}^{n} (-1)^{i} \binom{n}{i}\binom{n+k-i-2}{k-i-1}\right]$$

$$= -\left[\sum_{i=0}^{n} (-1)^{i} \binom{n}{i}\binom{n+k-i-2}{n-1} - \binom{n+k-2}{k-1}\right] .$$

But $\binom{n}{k} = \binom{n+1}{k+1} - \binom{n}{k+1}$. Performing this transformation, we have

$$M_k = -\left[\sum_{i=0}^{n} (-1)^{i} \binom{n}{i}\left\{\binom{n+k-i-1}{n} - \binom{n+k-i-2}{n}\right\} - \binom{n+k-2}{k-1}\right] .$$

Let $X = n+k-1$, and $Y = n+k-2$. Then,

$$M_k = \sum_{i=0}^{n} (-1)^{i} \binom{n}{i}\binom{Y-i}{n} - \sum_{i=0}^{n} (-1)^{i} \binom{n}{i}\binom{X-i}{n} + \binom{n+k-2}{k-1} .$$

By use of the following combinatorial identity:

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{Z-i}{n} = 1, \quad \text{for } Z > n,$$

which is derived in Appendix I, the desired solution is obtained.

$$M_k = 1 - 1 + \binom{n+k-2}{k-1} = \binom{n+k-2}{k-1}.$$

The number of ways n tests can sum to s successes,

$$N_s = \binom{n+s-1}{n-1}, \tag{2}$$

and the value for M_s,

$$M_s = \binom{n+s-2}{n-1}, \tag{3}$$

which was just proved, provides the derivation for $\hat{\theta}_s$.

$$\hat{\theta}_s = \frac{M_s}{N_s} = \frac{(n+s-2)!\ (n-1)!\ s!}{(n-1)!\ (s-1)!\ (n+s-1)!}.$$

$$\hat{\theta}_s = \begin{cases} \dfrac{s}{n+s-1} & \text{for } s > 0. \\ 0 & \text{for } s = 0. \end{cases} \tag{4}$$

Thus, $\hat{\theta}_s$ is equal to the number of successful cycles, divided by the total number of cycles minus one.

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3. Derivation of the Variance.

The variance of a random variable is defined by

$$\mathrm{Var}(X) = E(X^2) - E^2(X).$$

Therefore, the variance for the unbiased estimator $\hat{\theta}_s$ can be obtained by solving the following equation:

$$\mathrm{Var}(\hat{\theta}_s) = E(\hat{\theta}_s^2) - E^2(\hat{\theta}_s).$$

Equation (4) gives the value for $\hat{\theta}_s$ such that $E(\hat{\theta}_s) = \theta$. Therefore, $E^2(\hat{\theta}_s) = \theta^2$. However, since the derivation of $E(\hat{\theta}_s)$ provides a formula which is useful in deriving the second moment $E(\hat{\theta}_s^2)$, the reverse computation will be shown.

$$E(\hat{\theta}_s) = \sum_{s=0}^{\infty} \theta_s N_s \theta^s (1-\theta)^n = (1-\theta)^n \sum_{s=0}^{\infty} M_s \theta_s$$

$$= (1-\theta)^n \sum_{s=0}^{\infty} \binom{n+s-2}{s-1} \theta^s .$$

Since the s=0 term equals zero, the summation index can be changed, and

$$\sum_{s=0}^{\infty} \binom{n+s-2}{s-1} \theta^s = \theta \sum_{s=0}^{\infty} \binom{n-1+s}{s} \theta^s .$$

By use of the following identity (also derived in Appendix I), namely

$$\sum_{i=0}^{\infty} \binom{N+i}{i} \theta^i = \frac{1}{(1-\theta)^N} ,$$

the summation

$$\sum_{s=0}^{\infty} \binom{n-1+s}{s} \theta^s = \frac{1}{(1-\theta)^n} ,$$

and

$$\sum_{s=0}^{\infty} \binom{n+s-2}{s-1} \theta^s = \frac{\theta}{(1-\theta)^n} . \tag{6}$$

Thus,

$$E(\hat{\theta}_s) = \frac{(1-\theta)^n \theta}{(1-\theta)^n} = \theta , \text{ and } E^2(\hat{\theta}_s) = \theta^2. \tag{7}$$

Similarly, to find the second moment, we note that

$$E(\hat{\theta}_s^2) = \sum_{s=0}^{\infty} (\hat{\theta}_s)^2 N_s \theta^s (1-\theta)^n$$

$$= (1-\theta)^n \sum_{s=0}^{\infty} \left(\frac{s}{n+s-1}\right)^2 \binom{n+s-1}{n-1} \theta^s . \tag{8}$$

But $\left(\frac{s}{n+s-1}\right)^2 = \left(\frac{s}{n+s-1}\right)\left(\frac{s}{n+s-1} - 1\right) + \left(\frac{s}{n+s-1}\right)$

$$= \left(\frac{s}{n+s-1}\right)\left(\frac{1-n}{n+s-1}\right) + \left(\frac{s}{n+s-1}\right)$$

$$= -\left(\frac{s}{n+s-1}\right)\left(\frac{n-1}{n+s-1}\right) + \left(\frac{s}{n+s-1}\right) .$$

Substituting this expression into equation (8), and using equation (6), we obtain

$$E(\hat{\theta}_s^2) = (1-\theta)^n \sum_{s=0}^{\infty} \binom{n+s-2}{s-1} \theta^s - (1-\theta)^n \sum_{s=0}^{\infty} \left(\frac{n-1}{n+s-1}\right)\binom{n+s-2}{n-1} \theta^s$$

$$= \theta - (1-\theta)^n \sum_{s=0}^{\infty} \left(\frac{n-1}{n+s-1}\right)\binom{n+s-2}{n-1} \theta^s .$$

Multiplying by a factor $\frac{\theta^{n-1}}{\theta^{n-1}}$, we have

$$E(\hat{\theta}_s^2) = \theta - \frac{(1-\theta)^n}{\theta^{n-1}} \sum_{s=0}^{\infty} \left(\frac{n-1}{n+s-1}\right)\binom{n+s-2}{n-1} \theta^{n+s-1}$$

$$= \theta - \theta(n-1)\, R^{-n} \sum_{s=0}^{\infty} \frac{1}{n+s-1} \binom{n+s-2}{n-1} \theta^{n+s-1}$$

$$= \theta - \theta(n-1)\, R^{-n} f(\theta) , \tag{9}$$

where $f(\theta) = \sum_{s=0}^{\infty} \frac{1}{n+s-1} \binom{n+s-2}{n-1} \theta^{n+s-1}$. (10)

Differentiating equation (10), we have

$$f'(\theta) = g(\theta) = \sum_{s=0}^{\infty} \frac{n+s-1}{n+s-1} \binom{n+s-2}{n-1} \theta^{n+s-2}$$

$$= \theta^{n-2} \sum_{s=0}^{\infty} \binom{n+s-2}{n-1} \theta^{s} .$$

Thus, by substituting equation (6), we obtain

$$g(\theta) = \frac{\theta^{n-1}}{(1-\theta)^{n}} .$$

This expression must be integrated to obtain $f(\theta)$. Thus,

$$f(\theta) = \int g(\theta) + C = G(\theta) + C ,$$

where:

$$G(\theta) = \int \frac{\theta^{n-1}}{(1-\theta)^{n}} \, d\theta .$$

Integrating by parts, letting $u = \theta^{n-1}$ and $dv = (1-\theta)^{-n} d\theta$,

$$G(\theta) = \frac{1}{n-1}\left(\frac{\theta}{1-\theta}\right)^{n-1} - \frac{1}{n-2}\left(\frac{\theta}{1-\theta}\right)^{n-2} + \frac{1}{n-3}\left(\frac{\theta}{1-\theta}\right)^{n-3} - - -$$

$$+ (-1)^{n+1} \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)^{2} + (-1)^{n}\left(\frac{\theta}{1-\theta}\right) + (-1)^{n} \ln(1-\theta). \quad (11)$$

To find the constant C, equations (9) and (11) are evaluated at $\theta = 0$. The constant equals the difference $f(\theta) - G(\theta)$ for all θ.

$f(0) = 0$, and

$G(0) = \pm \ln(1) = 0$.

Thus the constant C is zero. Therefore,

$$f(\theta) = \frac{1}{n-1} R^{n-1} - \frac{1}{n-2} R^{n-2} + \frac{1}{n-3} R^{n-3} - - - + (-1)^n \frac{1}{3} R^3$$

$$+ (-1)^{n-1} \frac{1}{2} R^2 + (-1)^n R + (-1)^n \ln(1-\theta). \qquad (12)$$

Substituting equation (12) into (9) gives the solution for $E(\hat{\theta}_s^2)$.

$$E(\hat{\theta}_s^2) = \theta - \theta(n-1)\left[\frac{1}{n-1} R - \frac{1}{n-2} R^2 + \frac{1}{n-3} R^3 - - - - - \right.$$

$$\left. + (-1)^{n-1} \frac{1}{2} R^{n-2} + (-1)^n R^{n-1} + (-1)^n R^n \ln(1-\theta)\right]. \qquad (13)$$

Substituting equations (13) and (7) into equation (5) provides the derivation of the variance.

$$\mathrm{Var}(\hat{\theta}_s) = \theta\left\{(1-\theta) - (n-1)\left[\frac{1}{n-1} R - \frac{1}{n-2} R^2 + \frac{1}{n-3} R^3 - - - - \right.\right.$$

$$\left.\left. + (-1)^{n-1} \frac{1}{2} R^{n-2} + (-1)^n R^{n-1} + (-1)^n R^n \ln(1-\theta)\right]\right\}. \qquad (14)$$

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4. Numerical Analysis of Variance.

The equation (14) for $Var(\hat{\theta}_s)$ lends itself nicely to recursive numerical analysis. If the alternating series in the square brackets is considered, for various values of n, we have

$$n=2 \qquad R + R^2\ln(1-\theta) = R(1 + R\ln(1-\theta)),$$

$$n=3 \qquad \frac{1}{2}R - R^2 - R^3\ln(1-\theta) = R(\frac{1}{2} - R(1 + R\ln(1-\theta))),$$

$$n=4 \qquad \frac{1}{3}R - \frac{1}{2}R^2 + R^3 + R^4\ln(1-\theta) = R(\frac{1}{3} - R(\frac{1}{2} - R(1 + R\ln(1-\theta)))); \qquad (15)$$

and so forth. If RW_n is set equal the sum of the series in equations (15), Q is defined as -R, and $L = (1 + R\ln(1-\theta))$, then

$$W_2 = L,$$

$$W_3 = LQ + \frac{1}{2},$$

$$W_4 = (LQ + \frac{1}{2})Q + \frac{1}{3},$$

$$W_5 = ((LQ + \frac{1}{2})Q + \frac{1}{3})Q + \frac{1}{4}; \quad \text{etc.}$$

$Var(\hat{\theta}_s)$ can be re-written as

$$Var(\hat{\theta}_s) = \theta\left[(1-\theta) - (n-1)RW_n\right],$$

where W_n is defined by the following recursive formula,

$$W_2 = (1 + R\ln(1-\theta)), \quad \text{and}$$

$$W_n = W_{n-1}(-R) + \frac{1}{n-1}.$$

Appendix II provides tables of $\text{Var}(\hat{\Theta}_s)$ for 15 values of Θ and representative values of n between one and 50. Input data with nine-place accuracy was used in compiling these values.

Any output error (E_o) in this data is a result of the natural limitations on input data degrees of significance, and occurs in the bracket summation. We have

$$E_o = \; ----+(-1)^{n-1}(.6666---+\epsilon_2)\,R^{n-6} + (-1)^n(.2)\,R^{n-5}$$

$$+ (-1)^{n-1}(.25)\,R^{n-4} + (-1)^n(.333---+\epsilon_1)\,R^{n-3} + (-1)^n\,R^{n-1}$$

$$+ (-1)^n\,R^n\,(\ln(1-\Theta) + \epsilon_0) \quad .$$

Thus, for input data significance of degree ≥ 3,

$$E_o = (-1)^n\left[R^n\epsilon_0 + R^{n-3}\epsilon_1 + R^{n-6}\epsilon_2 + R^{n-7}\epsilon_3 + R^{n-9}\epsilon_4 \text{ -- etc.}\right] .$$

For values of $\Theta \geq .5$ (i.e., for values of $R \leq 1.0$), the computed variance has approximately the same degree of accuracy as the input data. For smaller values of Θ, however, E_o rapidly exceeds the value of the variance as n increases.

As an example, with an input accuracy to nine decimal places, $\Theta = 0.05$, $R = 19$, and $n = 6$,

$$E_o \sim (0.\epsilon_0 \times 10^{-9})(19^6) = (0.\epsilon_0 \times 10^{-9})(4.7\text{x}10^7) \gtrsim (4.7\text{x}10^{-3}) ,$$

but the variance at $n = 6$ is less than 9×10^{-3}. The tables of Appendix II indicate this decrease in accuracy for values of $\Theta < 0.5$.

Since Θ is normally larger than 0.5 for items that are cycled to failure, the inherent error term for small Θ is expected to place little restriction upon the use of the tables.

APPENDIX I

PROOF OF COMBINATORIAL IDENTITIES

A. To prove:

$$\sum_{i=0}^{n} (-1)^{i} \binom{n}{i}\binom{X-i}{n} = 1 \quad .$$

Each $\binom{X-i}{n}$ term can be written as a linear combination of $\binom{X-n}{j}$ terms, where j = 0, 1, 2, ---, n, by successive application of the rule:

$$\binom{X-i}{n} = \binom{X-i-1}{n} + \binom{X-i-1}{n-1} \quad .$$

Thus:

$$\binom{X}{n} = \binom{n}{0}\binom{X-n}{0} + \binom{n}{1}\binom{X-n}{1} + \binom{n}{2}\binom{X-n}{2} + - - - - + \binom{n}{n}\binom{X-n}{n}$$

$$\binom{X-1}{n} = \binom{n-1}{0}\binom{X-n}{1} + \binom{n-1}{1}\binom{X-n}{2} - - - - + \binom{n-1}{n-1}\binom{X-n}{n}$$

$$\binom{X-2}{n} = \binom{n-2}{0}\binom{X-n}{2} + - - - + \binom{n-2}{n-2}\binom{X-n}{n}$$

(etc., until finally:)

$$\binom{X-n}{n} = \binom{n-n}{n-n}\binom{X-n}{n} \quad .$$

Thus: $$\sum_{i=0}^{n} (-1)^{i} \binom{n}{i}\binom{n-i}{n} =$$

$$\binom{n}{0}\binom{n}{0}\binom{X-n}{0} + \binom{n}{0}\binom{n}{1}\binom{X-n}{1} + \binom{n}{0}\binom{n}{2}\binom{X-n}{2} + - - - - - + \binom{n}{0}\binom{n}{n}\binom{X-n}{n}$$

$$- \binom{n}{1}\binom{n-1}{0}\binom{X-n}{1} - \binom{n}{1}\binom{n-1}{1}\binom{X-n}{2} - - - - \binom{n}{1}\binom{n-1}{n-1}\binom{X-n}{n}$$

$$+ \binom{n}{2}\binom{n-2}{0}\binom{X-n}{2} + - - - + \binom{n}{2}\binom{n-2}{n-2}\binom{X-n}{n} ;$$

etc.,

(to the last term) $(-1)^n \binom{n}{n}\binom{n-n}{n-n}\binom{X-n}{n}$.

By summing on like coefficients, we have

$$1 + \sum_{r=1}^{n} \left[\sum_{i=0}^{r} (-1)^i \binom{n}{i} \binom{n-i}{r-i} \binom{X-n}{r} \right]$$

$$= 1 + \sum_{r=1}^{n} \binom{X-n}{r} \left[\sum_{i=0}^{r} (-1)^i \binom{n}{i}\binom{n-i}{r-i} \right] .$$

It will now be shown that the summation in the square brackets equals zero.

$$\binom{n}{i}\binom{n-i}{r-i} = \frac{n! \quad (n-i)!}{i! \ (n-i)! \ (r-i)! \ (n-r)!} \cdot \frac{r!}{r!} = \binom{r}{i}\binom{n}{r} .$$

Thus, the summation in the square brackets can be written as

$$\binom{n}{r} \sum_{i=0}^{r} (-1)^i \binom{r}{i} .$$

But this equals zero, as can be seen from

$$(x+y)^r = \binom{r}{0} x^r + \binom{r}{1} x^{r-1} y + - - - - - - - + \binom{r}{r} y^r .$$

Letting $x = 1$ and $y = -1$, the solution is provided.

$$\binom{r}{0} - \binom{r}{1} + \binom{r}{2} --- - - - + (-1)^r \binom{r}{r} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} = 0 .$$

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B. To prove:

$$\sum_{i=0}^{\infty} \binom{N+i}{N} \theta^i = \sum_{i=0}^{\infty} \binom{N+i}{i} \theta^i = \frac{1}{(1-\theta)^{N+1}} .$$

This proof can be seen from the successive differentiation of

$$\frac{1}{1-\theta} = \sum_{i=0}^{\infty} \theta^i ,$$

$$\frac{1}{(1-\theta)^2} = \sum_{i=0}^{\infty} (1 + i)\, \theta^i ,$$

$$\frac{1}{(1-\theta)^3} = \sum_{i=0}^{\infty} \binom{2+i}{2} \theta^i ;$$

etc. By induction,

$$\frac{1}{(1-\theta)^{N+1}} = \sum_{i=0}^{\infty} \binom{N+i}{N} \theta^i .$$

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APPENDIX II

TABLES OF VARIANCE

n	θ = .999	θ = .99	θ = .95	θ = .90	θ = .80
1	.00099900	.00990000	.04750000	.09000000	.16000000
2	.00000591	.00036517	.00538351	.01558428	.04047190
3	.00000099	.00009262	.00193332	.00653683	.01976405
4	(less than 10^{-6})	.00004860	.00109737	.00391053	.01258848
5		.00003268	.00075632	.00275400	.00913717
6		.00002459	.00057524	.00211750	.00714463
7		.00001970	.00046367	.00171767	.00585661
8		.00001643	.00038820	.00144401	.00495849
9		.00001410	.00033379	.00124521	.00429757
10		.00001234	.00029274	.00109435	.00379131
12		.00000988	.00023491	.00088071	.00306739
14		.00000824	.00019614	.00073676	.00257505
16		.00000706	.00016835	.00063321	.00221867
18		.00000618	.00014746	.00055516	.00194882
20		.00000549	.00013118	.00049423	.00173744
25		.00000430	.00010279	.00038780	.00136670
30		.00000353	.00008451	.00031907	.00112630
35		.00000300	.00007175	.00027107	.00095780
40		.00000260	.00006233	.00023557	.00083315
45		.00000230	.00005510	.00020831	.00073720
50		.00000206	.00004937	.00018671	.00066107

n	θ = .70	θ = .60	θ = .50	θ = .40	θ = .30
1	.21000000	.24000000	.25000000	.24000000	.21000000
2	.06479650	.08434420	.09657359	.09974306	.09256908
3	.03446014	.04754107	.05685282	.06077082	.05801098
4	.02284705	.03245893	.03972077	.0432656	.0419615
5	.01694454	.02448095	.03037231	.0334686	.0327862
6	.01342257	.01959921	.02453463	.0272462	.0268734
7	.01109697	.01632064	.02055846	.0229567	.022754
8	.00945152	.01397284	.01768180	.019825	.019723
9	.00822783	.01221117	.01550652	.017441	.017402
10	.00728301	.01084162	.01380517	.015567	.01556
12	.00592068	.00885224	.01131743	.012809	.0128
14	.00498649	.00747794	.00958726	.01087	.0109
16	.00430640	.00647221	.00831498	.00945	.009
18	.00378929	.00570452	.00734031	.00835	
20	.00338292	.00509941	.00656988	.0074	
25	.00266745	.00403018	.00520383	.0059	
30	.00220165	.00333140	.00430779	.004	
35	.00187430	.00283904	.00367488		
40	.00163166	.00247343	.00320407		
45	.00144463	.00219123	.00284018		
50	.00129606	.00196682	.00255049		

n	θ = .20	θ = .10	θ = .05	θ = .03	θ = .01
1	.16000000	.09000000	.04750000	.02910000	.00990000
2	.07405936	.0434201	.0233439	.0144022	.00493
3	.0475250	.028436	.015429	.00955	.003
4	.034849	.02110	.01151	.007	
5	.02746	.0167	.0091		
6	.02265	.013	.007		
7	.0192	.01			
8	.0167				
9	.014				
10	.013				
12	.01				
14					
16					
18					
20					

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